Univerzitet u Zenici
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Odsjeci: Inžinjerski dizajn proizvoda, Inžinjerska ekologija,Menadžment proizvodnim tehnologijama, Održavanje
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Parcijalni ispit iz predmeta **Matematike II**

1. Izračunati površinu figure koju čine linije $y=(x-1)^2$, $\frac{x^2}{1}-\frac{y^2}{2}=1$.
2. Izračunati površinu figure koju čine linije $y=\ln(x+2)$, $y=2\ln x$, $y=0$.
3. Naći ekstreme funkcije $z=\frac{8}{x}+\frac{x^2}{y}+y+1$.
4. Naći ekstreme funkcije $z=x+y-\frac{3}{2}\ln\left(x^2+y^2+1\right)$.
5. Izračunati dvostruki integral $I=\iint_D xy\,dx\,dy$, $D: y=\ln x,\ x=2,\ x+y=1$.
6. Izračunati dvostruki integral: $I=\iint_D x^2y^2\sqrt{\left(x^2+y^2\right)^3+1}\,dx\,dy$, $D: x^2+y^2\le 2,\ 0\le y\le x\sqrt{3}$.
7. Izračunati zapreminu tijela koje je ograničeno cilindrom $y=2x^2$ i ravnima $y+z=8,\ z=0$.
8. Izračunati zapreminu tijela koje je ograničeno površima $z^2=x^2+y^2, x^2+y^2+z^2=4$.

(Rješenja zadataka su skinuta sa stranice: pf.unze.ba\nabokov\
Za uočene greške pisati na: **infoarrt@gmail.com**)

Izračunati površinu figure koju čine linije

$$y=(x-1)^2, \quad \frac{x^2}{1}-\frac{y^2}{2}=1.$$

Rj: Da bi odredili granice za računanje površine potrebno je grafički predstaviti ove dvije linije.

ispitajmo f-ju $y=(x-1)^2$

$D: x\in\mathbb{R}$

f-ja nije ni parna ni neparna

$f(0)=1$, $(0,1)$ je presjek sa y-osom

$(x-1)^2=0 \Rightarrow x=1$ $(1,0)$ je nula f-je

$y=(x-1)^2=x^2-2x+1 \Rightarrow$ f-ja je oblika 

Nađimo još tjeme f-je

$y'=2x-2$

$y'=0 \Rightarrow 2x-2=0 \Rightarrow x=1$

$T(1,0)$

Prije nego grafički predstavimo liniju $y=(x-1)^2$ pronađimo u kojim tačkama siječe liniju $\frac{x^2}{1}-\frac{y^2}{2}=1$.

$$\begin{array}{l} y=x^2-2x+1=(x-1)^2 \\ 2x^2-y^2=2 \\ \hline \end{array}$$

$$y^2=(x^2-2x+1)^2=$$
$$=(x^2-2x+1)(x^2-2x+1)=$$
$$=x^4-2x^3+x^2-2x^3+4x^2-2x+x^2-2x+1$$
$$=x^4-4x^3+6x^2-4x+1$$

$$\begin{array}{l} 2x^2-y^2=2 \\ \hline 2x^2-x^4+4x^3-6x^2+4x-1-2=0 \quad /\cdot(-1) \\ x^4-4x^3+4x^2-4x+3=0 \end{array}$$

Označimo ovo sa $g(x)=x^4-4x^3+4x^2-4x+3$

Kako je $g(1)=0 \Rightarrow g(x)$ je djeljivo sa $(x-1)$

$$(x^4-4x^3+4x^2-4x+3):(x-1)=x^3-3x^2+x-3$$
$$\begin{array}{l} -\;x^4-x^3 \\ \hline \quad -3x^3+4x^2-4x+3 \\ \quad -\;-3x^3+3x^2 \\ \hline \qquad x^2-4x+3 \\ \qquad -\;x^2-x \\ \hline \qquad\quad -3x+3 \\ \qquad\quad -3x+3 \\ \hline \qquad\quad = \;\; = \end{array}$$

$$g(x)=\underbrace{(x^3-3x^2+x-3)}_{g_1(x)}(x-1)$$

$g_1(0)=-3$

$g_1(1)=1-3+1-3=-4$

$g_1(2)=8-12+2-3=-5$

$g_1(3)=27-27+3-3$

$g_1(-2)=-8-12-2-3=-25$

$g_1(-1)=-1-3-1-3=-8$

$g_1(-3)=-27-27-3-3<0$

$\Rightarrow g_1(x)$ je djeljivo sa $x-3$

$$(x^3-3x^2+x-3):(x-3)=x^2+1$$
$$\begin{array}{l} -\;x^3-3x^2 \\ \hline \qquad x-3 \\ \qquad -\;x-3 \\ \hline \qquad = \;\; = \end{array}$$

Prema tome $g(x)=(x^2+1)(x-3)(x-1)$

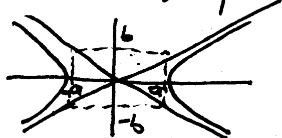
Za $x=3 \Rightarrow y=4$

Za $x=1 \Rightarrow y=0$

Presječne tačke krivih su $(3,4)$ i $(1,0)$

Krive oblika

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ zovemo HIPERBOLE i one su oblika

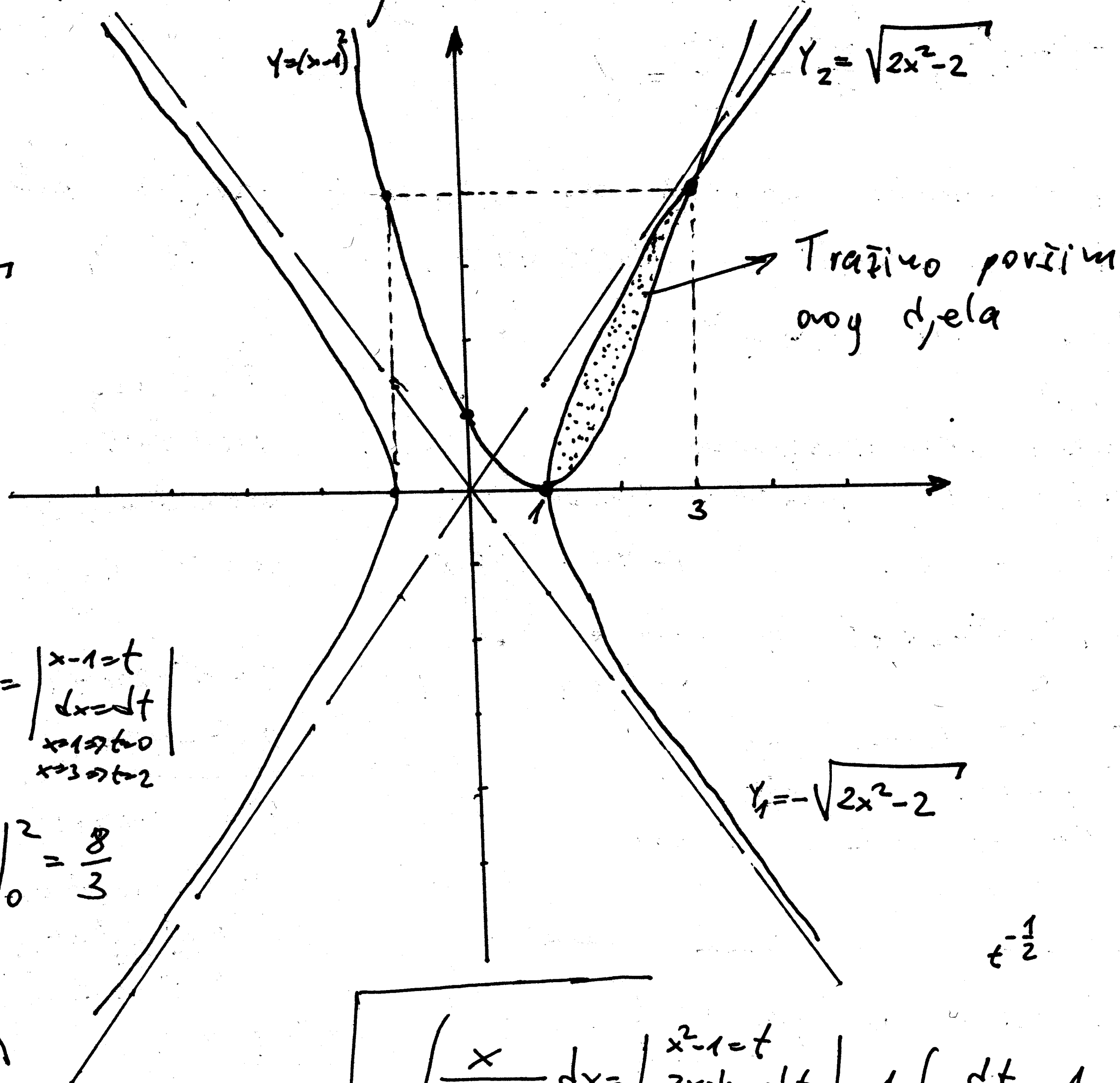


Nacrtajmo naše krive linije

$\sqrt{2} \approx 1,41$

$y^2 = 2x^2 - 2$

$y_{1,2} = \pm\sqrt{2x^2-2}$



$$P_2 = \int_1^3 (x-1)^2\,dx = \left| \begin{matrix} x-1=t \\ dx=dt \\ x=1 \Rightarrow t=0 \\ x=3 \Rightarrow t=2 \end{matrix} \right|$$

$$= \int_0^2 t^2\,dt = \frac{1}{3}t^3\Big|_0^2 = \frac{8}{3}$$

$t^{-\frac{1}{2}}$

$$\boxed{\int \frac{x}{\sqrt{x^2-1}}\,dx = \left| \begin{matrix} x^2-1=t \\ 2x\,dx=dt \\ x\,dx=\frac{1}{2}dt \end{matrix} \right| = \frac{1}{2}\int \frac{dt}{\sqrt{t}} = \frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = \sqrt{t} + C = \sqrt{x^2-1} + C}$$

$$P = \int_1^3 \Big(\underbrace{\sqrt{2x^2-2}}_{P_1} - \underbrace{(x-1)^2}_{P_2}\Big)\,dx$$

$$P_1 = \int_1^3 \sqrt{2}\cdot\sqrt{x^2-1}\,dx = \sqrt{2}\int_1^3 \frac{x^2-1}{\sqrt{x^2-1}}\,dx = \sqrt{2}\left(\int_1^3 \frac{x^2}{\sqrt{x^2-1}}\,dx - \int_1^3 \frac{dx}{\sqrt{x^2-1}}\right)$$

$$\int_1^3 x\cdot\frac{x}{\sqrt{x^2-1}}\,dx = \left| \begin{matrix} u=x & dv=\frac{x}{\sqrt{x^2-1}}\,dx \\ du=dx & v=\sqrt{x^2-1} \end{matrix} \right| = \underbrace{x\sqrt{x^2-1}\Big|_1^3}_{3\sqrt{8}-0} - \int_1^3 \sqrt{x^2-1}\,dx$$

$$\int_1^3 \frac{dx}{\sqrt{x^2-1}} = \ln\left|x+\sqrt{x^2-1}\right|\Big|_1^3 = \ln|3+\sqrt{8}|$$

$$\sqrt{2}\int_1^3 \sqrt{x^2-1}\,dx = \underbrace{\sqrt{2}\cdot 3\sqrt{8}}_{6\sqrt{2}} - \sqrt{2}\int_1^3 \sqrt{x^2-1}\,dx - \sqrt{2}\ln(3+2\sqrt{2})$$

$$\int_1^3 \sqrt{2x^2-2}\,dx = 6 - \frac{\sqrt{2}\ln(2\sqrt{2}+3)}{2}$$

$$P = P_1 - P_2 = \frac{10}{3} - \frac{\sqrt{2}\ln(2\sqrt{2}+3)}{2}$$

tražena površina

(#) Izračunati površinu figure koju čine linije $y=\ln(x+2)$, $y=2\ln x$, $y=0$.

Rj. Grafički predstavimo f-ju $y=\ln(x+2)$

definiciono područje $\mathcal{D}: x+2>0$

$x>-2$

f-ja nije ni parna ni neparna

$y=0 \Rightarrow \ln(x+2)=0$

$x+2=1$

$x=-1$

$(-1,0)$ je nula f-je

$x=0 \Rightarrow y=\ln 2$

$\ln 2 \approx 0{,}69$

$(0,\ln 2)$ je presek sa y-osom

$y<0$

$\ln(x+2)<0$

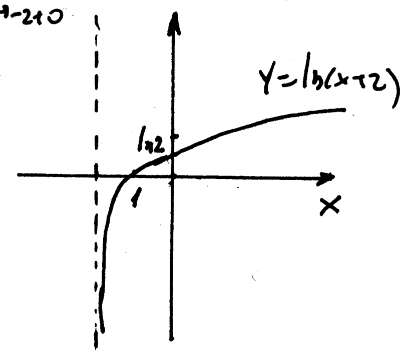
$\ln(x+2)<\ln 1$

$x<-1$

x	$(-\infty,-1)$	$(-1,+\infty)$
y	−	+

znak f-je

$\lim\limits_{x\to -2+0} \ln(x+2)=\ln(+0)=-\infty$



Pronađimo presječne tačke krivih $y=\ln(x+2)$ i $y=2\ln x$.

$y=\ln(x+2)$

$y=2\ln x=\ln x^2$

$\ln(x+2)=\ln x^2$

$x+2=x^2$

$x^2-x-2=0$

$(x+1)(x-2)=0$

$x_1=-1 \Rightarrow$
$\Rightarrow y=\ln 1=0$

$x_2=2 \Rightarrow$
$\Rightarrow y=\ln 4\approx 1{,}38$

Presječne tačke krivih su $(-1,0)$ i $(2,\ln 4)$

Grafički predstavimo f-ju $y=\ln x^2$

definiciono područje $\mathcal{D}: x\neq 0$

$f(-x)=\ln(-x)^2=\ln x^2=f(x) \Rightarrow$ f-ja $y=2\ln x$ je parna (simetrična je u odnosu na y-osu)

$x=0 \Rightarrow \ln x^2=0$

$x_{1,2}=\pm 1$

$\Rightarrow (-1,0)$ i $(1,0)$ su nule f-je

$\lim\limits_{x\to 0+} \ln x^2=\ln(0+)=-\infty \Rightarrow x=0$ je V. A.

$\lim\limits_{x\to +\infty} \ln x^2=+\infty$

$y<0$

$\ln x^2<0$

$\ln x^2<\ln 1$

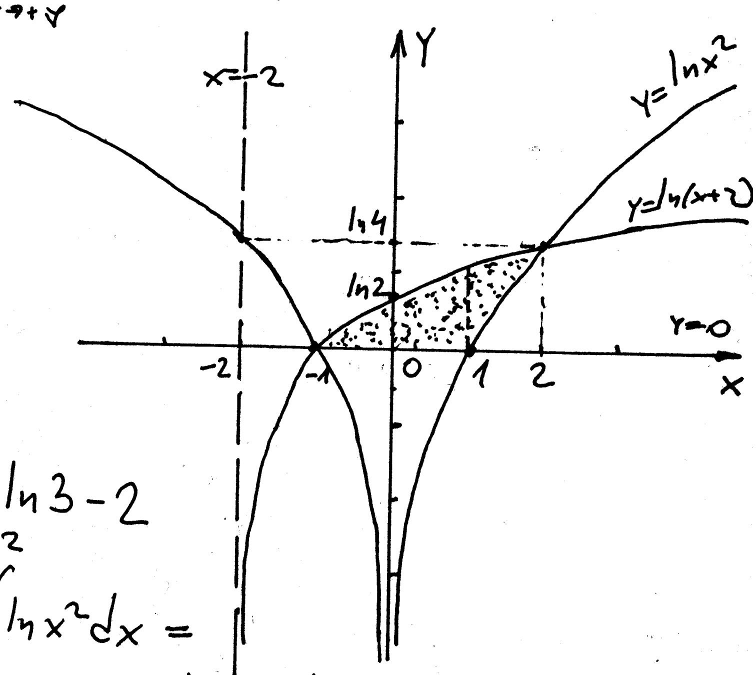
$x^2<1$

$x\in(-1,1)$ znak f-je

x	$(-\infty,-1)$	$(-1,0)$	$(0,1)$
y	+	−	−

x	$(1,+\infty)$
y	+

znak f-je



Izračunajmo površinu $P=P_1+P_2$

$$P_1=\int_{-1}^{1}\ln(x+2)\,dx=\begin{vmatrix} u=\ln(x+2) & dv=dx \\ du=\dfrac{dx}{x+2} & v=x \end{vmatrix}=$$

$$=x\ln(x+2)\Big|_{-1}^{1}-\int_{-1}^{1}\frac{x+2-2}{x+2}\,dx=\ldots=3\ln 3-2$$

$$P_2=\int_{1}^{2}\left[\ln(x+2)-\ln x^2\right]dx=\int_{1}^{2}\ln(x+2)\,dx-\int_{1}^{2}\ln x^2\,dx=$$

$$=\ldots=4\ln 2-3\ln 3+1$$

$$\int \ln x\,dx=\begin{vmatrix} u=\ln x & dv=dx \\ du=\dfrac{dx}{x} & v=x \end{vmatrix}=x\ln x-\int dx=x\ln x-x+C$$

$\Rightarrow P=4\ln 2-1$ tražena površina

(#) Naći ekstreme f-je $z=\frac{8}{x}+\frac{x^2}{y}+y+1.$

Rj. Pronađimo stacionarne tačke

$$\frac{\partial z}{\partial x}=8\cdot(-1)x^{-2}+2\frac{x}{y}=\frac{-8}{x^2}+2\frac{x}{y}$$

$$\frac{\partial z}{\partial y}=x^2\cdot(-1)y^{-2}+1=\frac{-x^2}{y^2}+1$$

$$-\frac{8}{x^2}+\frac{2x}{y}=0$$

$$-\frac{x^2}{y^2}+1=0$$

$$\frac{8}{x^2}-2\frac{x}{y}=0.$$

$$\frac{x^2}{y^2}=1 \Rightarrow \left(\frac{x}{y}\right)^2=1$$

Prema tome $\frac{x}{y}=1$ i $\frac{x}{y}=-1$

Za $\frac{x}{y}=1 \Rightarrow \frac{8}{x^2}-2\cdot1=0$

$$\frac{8}{x^2}=2 \quad /\cdot x^2 \ (x\neq0)$$

$2x^2=8$

$x^2=4$

$x_1=-2,\ x_2=2$

$x_1=-2 \Rightarrow \frac{x}{y}=1$

$y=-2$

$(-2,-2)$

za $x_2=2 \Rightarrow$

$\Rightarrow \frac{x}{y}=1$

$y_2=2$

$(2,2)$

Za $\frac{x}{y}=-1$ imamo

$$\frac{8}{x^2}+2=0$$

$$\frac{8}{x^2}=-2 \quad /\cdot x^2 \ (x\neq0)$$

$-2x^2=8$

ova eduačina nema rješenja u skupu realnih brojeva

Stacionarne tačke su $M_1(-2,-2)$ i $M_2(2,2)$.

Nađimo druge parcijalne izvode

$$\frac{\partial^2 z}{\partial x^2}=(-8)(-2)x^{-3}+\frac{2}{y}=\frac{16}{x^3}+\frac{2}{y}$$

$$\frac{\partial^2 z}{\partial x\partial y}=2x\cdot(-1)y^{-2}=\frac{-2x}{y^2}$$

$$\frac{\partial^2 z}{\partial y^2}=-x^2\cdot(-2)y^{-3}=\frac{2x^2}{y^3}$$

Za $M_2(2,2)$

$A=2+1=3,\quad B=\frac{-4}{4}=-1,\quad C=\frac{8}{8}=1$

$D=AC-B^2=3-1=2>0$ f-ja ima ekstrem

$A>0 \Rightarrow$ f-ja ima minimum

$z_{min}(2,2)=4+2+2+1=9$

Za $M_1(-2,-2)$

$A=\frac{16}{-8}+\frac{2}{-2}=-2-1=-3$

$B=\frac{-2\cdot(-2)}{4}=1,\quad C=\frac{2\cdot4}{-8}=-1$

$D=AC-B^2=3-1=2>0$

f-ja u tački $M_1(-2,-2)$ ima ekstrem

$A<0$ f-ja ima maksimum

$z_{max}(-2,-2)=-4-2-2+1=-7$

#) Naći ekstreme f-je $z = x + y - \frac{3}{2}\ln(x^2+y^2+1)$.

Rj: Pronađimo prve parcijalne izvode

$$\frac{\partial z}{\partial x} = 1 - \frac{3}{2}\cdot\frac{2x}{x^2+y^2+1} = 1 - \frac{3x}{x^2+y^2+1}$$

$$\frac{\partial z}{\partial y} = 1 - \frac{3}{2}\cdot\frac{2y}{x^2+y^2+1} = 1 - \frac{3y}{x^2+y^2+1}$$

Pronađimo stacionarne tačke

$$\left.\begin{aligned}\frac{\partial z}{\partial x} = 0 &\Rightarrow 1 = \frac{3x}{x^2+y^2+1}\\ \frac{\partial z}{\partial y} = 0 &\Rightarrow 1 = \frac{3y}{x^2+y^2+1}\end{aligned}\right\} \Rightarrow x = y \text{ (dijeljenjem jednačina)}$$

Sad imamo $x = y$ i $1 = \frac{3x}{x^2+y^2+1} \Rightarrow 1 = \frac{3x}{2x^2+1} \Rightarrow 2x^2-3x+1=0$

$D = 9-8 = 1$

$x_1 = 1, \ x_2 = \frac{1}{2}$

Stacionarne tačke su $M_1(1,1)$ i $M_2(\frac{1}{2}, \frac{1}{2})$.

Pronađimo druge parcijalne izvode.

$$\frac{\partial^2 z}{\partial x^2} = \left(1 - \frac{3x}{x^2+y^2+1}\right)'_x = \frac{-3(x^2+y^2+1)+3x\cdot 2x}{(x^2+y^2+1)^2} = \frac{3x^2-3y^2-3}{(x^2+y^2+1)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \left(1 - \frac{3y}{x^2+y^2+1}\right)'_y = \left|\begin{matrix}\text{zbog}\\ \text{simetričnosti}\end{matrix}\right| = \frac{3y^2-3x^2-3}{(x^2+y^2+1)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{3x\cdot 2y}{(x^2+y^2+1)^2} = \frac{6xy}{(x^2+y^2+1)^2}$$

Za tačku $M_1(1,1)$: $A = -\frac{3}{9} = -\frac{1}{3}$, $B = \frac{6}{9} = \frac{2}{3}$, $C = -\frac{3}{9} = -\frac{1}{3}$, $D = AC - B^2$

$D = \frac{1}{9} - \frac{4}{9} < 0 \Rightarrow$ u M_1 f-ja nema ekstrema

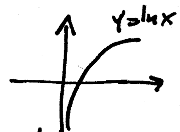
Za tačku $M_2(\frac{1}{2}, \frac{1}{2})$: $A = \frac{-3}{(\frac{3}{2})^2} = -\frac{3}{\frac{9}{4}} = -\frac{12}{9} = -\frac{4}{3} \Rightarrow C = -\frac{4}{3}$

$B = \frac{\frac{3}{2}}{\frac{9}{4}} = \frac{12}{18} = \frac{2}{3}$, $D = AC - B^2 = \frac{16}{9} - \frac{4}{9} = \frac{12}{9} = \frac{4}{3} > 0 \Rightarrow$ f-ja u tački M_2 ima ekstrem

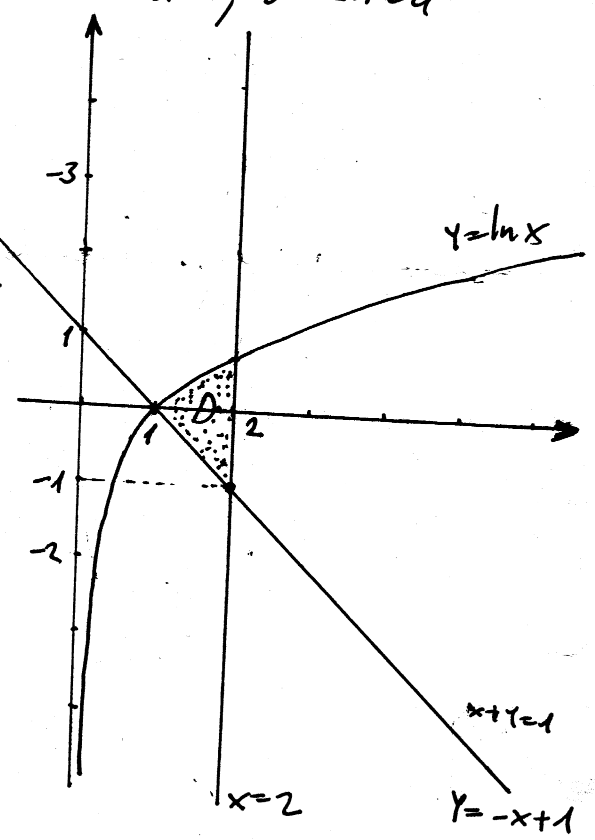
$A < 0 \Rightarrow$ u M_2 f-ja ima maximum $z_{max}(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2} + \frac{1}{2} - \frac{3}{2}\ln(\frac{1}{4}+\frac{1}{4}+1) = 1 - \ln\frac{27}{8}$

(#) Izračunati dvostruki integral $I=\iint\limits_D xy\,dx\,dy$, gdje je D: $y=\ln x$, $x=2$, $x+y=1$.

Rj. Kriva $y=\ln x$ izgleda ovako



Nacrtajmo sliku



Pronađimo presječne tačke datih krivi.

$$\begin{array}{c} y=\ln x \\ x=2 \\ \hline y=\ln 2 \approx 0,69 \\ (2,\ln 2) \end{array} \qquad \begin{array}{c} y=\ln x \\ x+y=1 \\ \hline y=\ln x \\ y=-x+1 \\ \hline \ln x=-x+1 \\ x=1 \\ (1,0) \end{array} \qquad \begin{array}{c} x=2 \\ x+y=1 \\ \hline 2+y=1 \\ y=-1 \\ (2,-1) \end{array}$$

$$I=\iint\limits_D xy\,dx\,dy=\int_1^2 dx\int_{-x+1}^{\ln x} xy\,dy=\int_1^2 x\,dx\int_{-x+1}^{\ln x} y\,dy=$$

$$=\int_1^2 x\left(\frac{1}{2}y^2\Big|_{-x+1}^{\ln x}\right)dx=\frac{1}{2}\int_1^2 x\left(\ln^2 x-\underbrace{(-x+1)^2}_{x^2-2x+1}\right)dx=$$

$$=\frac{1}{2}\int_1^2 x\ln^2 x\,dx-\frac{1}{2}\int_1^2 (x^3-2x^2+x)\,dx$$

$$\int_1^2 x\ln^2 x\,dx=\left|\begin{array}{ll} u=\ln^2 x & dv=x\,dx \\ du=2\ln x\cdot\frac{1}{x}\,dx & v=\frac{1}{2}x^2 \end{array}\right|=\overbrace{\frac{1}{2}x^2\ln^2 x\Big|_1^2}^{\substack{2\cdot 0,48\approx 0,96 \\ \frac{1}{2}4(\ln 2)^2}}-\int_1^2 x\ln x\,dx=$$

$$=\left|\begin{array}{ll} u=\ln x & dv=x\,dx \\ du=\frac{1}{x}dx & v=\frac{1}{2}x^2 \end{array}\right|=2\ln^2 2-\Bigg[\overbrace{\frac{1}{2}x^2\ln x\Big|_1^2}^{2\ln 2}-\frac{1}{2}\underbrace{\int_1^2 x\,dx}_{\substack{\frac{1}{2}x^2|_1^2 \\ \frac{1}{2}\cdot 3}}\Bigg]=2\ln^2 2-2\ln 2+\frac{3}{4}$$

$$\int_1^2 (x^3-2x^2+x)\,dx=\frac{1}{4}x^4\Big|_1^2-\frac{2}{3}x^3\Big|_1^2+\frac{1}{2}x^2\Big|_1^2=$$

$$=\frac{15}{4}-\frac{14}{3}+\frac{3}{2}=\frac{45-56+18}{12}=\frac{7}{12}$$

traženo rješenje

$$I=\frac{1}{2}\left(2\ln^2 2-2\ln 2+\frac{3}{4}\right)-\frac{1}{2}\cdot\frac{7}{12}=\ln^2 2-\ln 2+\frac{3}{8}-\frac{7}{24}=\ln^2 2-\ln 2+\frac{1}{12}$$

(#) Izračunati dvostruki integral

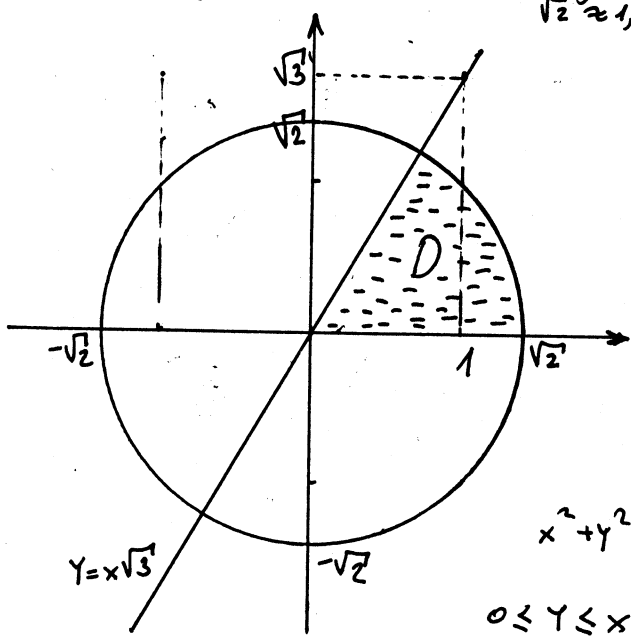
$$I=\iint\limits_{D} x^2y^2\sqrt{(x^2+y^2)^3+1}\,dx\,dy,\quad D:\ x^2+y^2\le 2,\ 0\le y\le x\sqrt{3}.$$

Rj. Nacrtajmo oblast integracije D

$\sqrt{2}\approx 1{,}41$

$y=x\sqrt{3}$

$x=1 \Rightarrow y=\sqrt{3}\approx 1{,}73$



Uvedimo polarne koordinate

$x=r\cos\varphi$

$y=r\sin\varphi$

$dx\,dy=r\,dr\,d\varphi$

$x^2+y^2=r^2\cos^2\varphi+r^2\sin^2\varphi=r^2$

$D \xrightarrow{\text{transformacije}} D'$

$$D':\begin{cases}0\le r\le\sqrt{2}\\ 0\le\varphi\le\frac{\pi}{3}\end{cases}$$

$x^2+y^2\le 2 \Rightarrow r^2\le 2$

$0\le y\le x\sqrt{3}$

$0\le r\sin\varphi\le r\cos\varphi\sqrt{3}\quad /:r$

$0\le\sin\varphi\le\cos\varphi\sqrt{3}$

$x^2y^2=r^2\cos^2\varphi\, r^2\sin^2\varphi$
$=r^4\cos^2\varphi\sin^2\varphi$

$$\left.\begin{array}{l}\sin\varphi\ge 0\\ \quad\varphi\in[0,\pi]\\ 0\le\sin\varphi\le\cos\varphi\sqrt{3}\quad /:\cos\varphi\\ 0\le \operatorname{tg}\varphi\le\sqrt{3}\\ \quad\varphi\in[0,\frac{\pi}{3}]\cup[\pi,\frac{4\pi}{3}]\end{array}\right\}\Rightarrow \varphi\in[0,\tfrac{\pi}{3}]$$

$1=\sin^2\varphi+\cos^2\varphi$

$\cos 2\varphi=\cos^2\varphi-\sin^2\varphi$

$\cos^2\varphi=\frac{1+\cos 2\varphi}{2}\qquad \sin^2\varphi=\frac{1-\cos 2\varphi}{2}$

$$I=\iint\limits_{D}x^2y^2\sqrt{(x^2+y^2)^3+1}\,dx\,dy=\iint\limits_{D'}r^4\cos^2\varphi\sin^2\varphi\sqrt{r^6+1}\,r\,dr\,d\varphi=$$

$$=\int_0^{\sqrt{2}}r^5\sqrt{r^6+1}\,dr\int_0^{\pi/3}\cos^2\varphi\sin^2\varphi\,d\varphi=\left(\frac{\sqrt{3}}{64}+\frac{\pi}{24}\right)\int_0^{\sqrt{2}}r^5\sqrt{r^6+1}\,dr=\underline{\underline{\frac{26}{9}\left(\frac{\sqrt{3}}{64}+\frac{\pi}{24}\right)}}$$

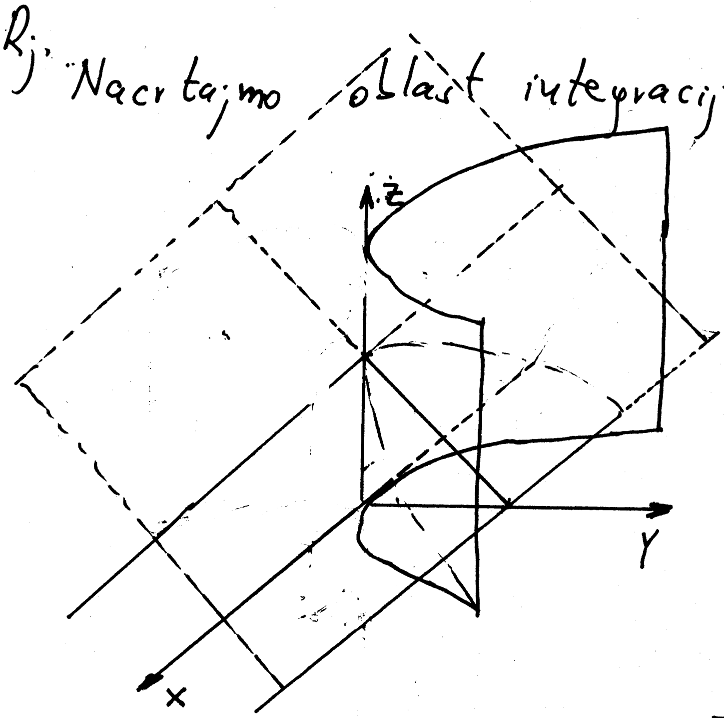
$$\int_0^{\sqrt{2}}r^5\sqrt{r^6+1}\,dr=\left|\begin{array}{ll}r^6+1=t & r=0\Rightarrow t=1\\ 6r^5dr=dt & r=\sqrt{2}\Rightarrow t=9\end{array}\right|=\frac{1}{6}\int_1^9\sqrt{t}\,dt=\frac{1}{6}\cdot\left.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right|_1^9=\frac{\cancel{2}}{\cancel{6}\cdot 3}\left.\sqrt{t^3}\right|_1^9=$$
$$=\frac{26}{9}$$

$$\int_0^{\pi/3}\cos^2\varphi\sin^2\varphi\,d\varphi\,d\varphi=\int_0^{\pi/3}\frac{1}{2}(1+\cos 2\varphi)\cdot\frac{1}{2}(1-\cos 2\varphi)\,d\varphi=\ldots=\frac{\sqrt{3}}{64}+\frac{\pi}{24}$$

tražena zapremina

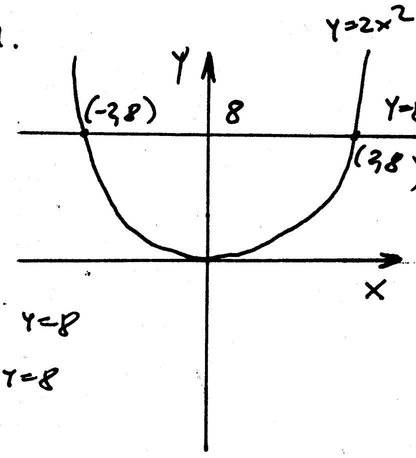
(#) Izračunati zapreminu tijela koje je ograničeno cilindrom $y=2x^2$ i ravnima $y+z=8$, $z=0$.

Rj. Nacrtajmo oblast integracije $\Omega: \begin{cases} y=2x^2 \\ y+z=8 \\ z=0 \end{cases}$



Ravan $y+z=8$ siječe cilindar

Napravimo projekciju oblasti Ω na xOy ravan.



Nađimo presjek krive $y=2x^2$ i prave $y=8$.

$$\begin{array}{l} y=2x^2 \\ y=8 \\ \hline x^2=4 \\ x_1=-2,\ x_2=2 \end{array} \qquad \begin{array}{l} x_1=-2 \Rightarrow y=8 \\ x_2=2 \Rightarrow y=8 \end{array}$$

$$\Omega: \begin{cases} -2 \le x \le 2 \\ 2x^2 \le y \le 8 \\ 0 \le z \le 8-y \end{cases} \qquad V=\iiint_\Omega dx\,dy\,dz$$

$$V=\iiint_\Omega dx\,dy\,dz = \int_{-2}^{2} dx \int_{2x^2}^{8} dy \int_{0}^{8-y} dz = \int_{-2}^{2} dx \int_{2x^2}^{8} z\Big|_0^{8-y} dy = \int_{-2}^{2} dx \int_{2x^2}^{8} (8-y)\,dy =$$

$$= \int_{-2}^{2} \left(8y\Big|_{2x^2}^{8} - \frac{1}{2}y^2\Big|_{2x^2}^{8}\right) dx = \int_{-2}^{2} \left[8(8-2x^2) - \frac{1}{2}(8^2-4x^4)\right] dx =$$

$$= \int_{-2}^{2} (64-16x^2-32+2x^4)\,dx = \int_{-2}^{2} (2x^4-16x^2+32)\,dx =$$

$$= 2\cdot\frac{1}{5}x^5\Big|_{-2}^{2} - 16\cdot\frac{1}{3}x^3\Big|_{-2}^{2} + 32x\Big|_{-2}^{2} = \frac{2}{5}\cdot 64 - \frac{16}{3}\cdot 16 + 32\cdot 4 =$$

$$= \frac{384-1280+1920}{15} = \frac{1024}{15}$$

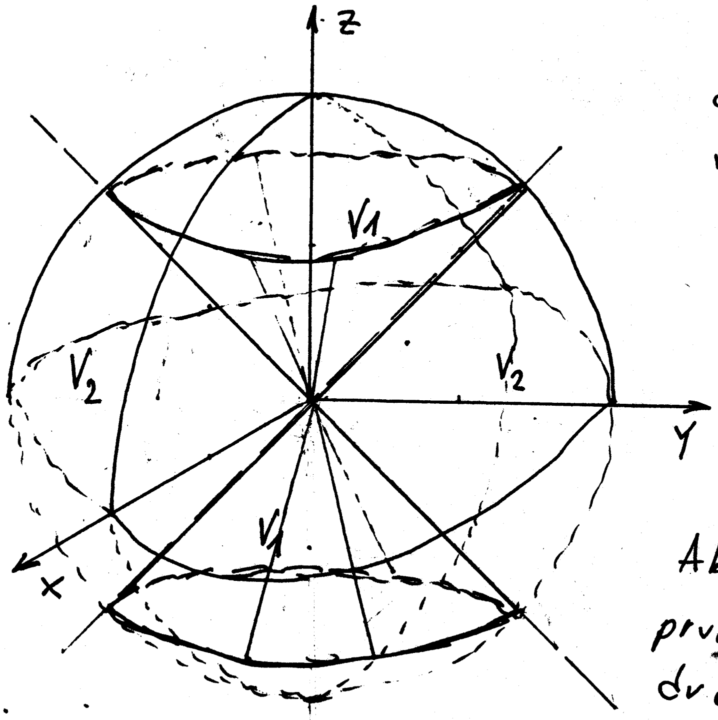
Izračunati zapreminu tijela koje je ograničeno površima $z^2=x^2+y^2$, $x^2+y^2+z^2=4$.

Rj.

$x^2+y^2+z^2=4$ je kugla sa centrom u $(0,0,0)$ poluprečnika $r=2$

$z^2=x^2+y^2$ je konus

Skicirajmo ove dvije figure u prostoru.



Presjek konusa i kugle daje dva tijela za koje možemo računati zapreminu: prvo tijelo je određeno u presjeku unutrašnjosti konusa i kugle, a drugo tijelo je određeno dijelom lopte van konusa. Ako sa V_1 označimo zapreminu prvog, a sa V_2 zapreminu drugog tijela, imamo da je

$$V=V_1+V_2=\frac{4}{3}r^3\pi \quad (\text{zapremina kugle})$$

Kako je $r=2 \Rightarrow V=\frac{4}{3}\cdot 8\pi=\frac{32\pi}{3}$.

$V=\iiint\limits_{\Omega} dx\,dy\,dz$ – zapremina tijela ograničenog sa oblasti Ω

Uvedimo sferne koordinate

$$x=\rho\sin\varphi\cos\theta \qquad z^2=x^2+y^2$$

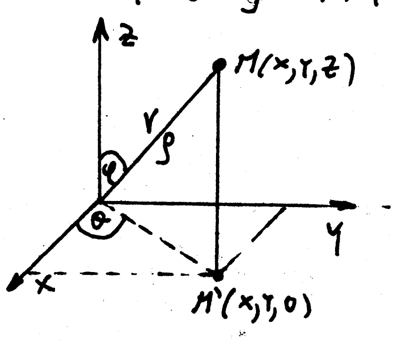
$$y=\rho\sin\varphi\sin\theta \qquad \rho^2\cos^2\varphi=\rho^2\sin^2\varphi\cos^2\theta+\rho^2\sin^2\varphi\sin^2\theta=$$

$$z=\rho\cos\varphi \qquad =\rho^2\sin^2\varphi(\cos^2\theta+\sin^2\theta)=\rho^2\sin^2\varphi$$

$$\Rightarrow \cos^2\varphi=\sin^2\varphi \quad /:\sin^2\varphi$$

$$dx\,dy\,dz=\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$tg^2\varphi=1 \Rightarrow tg\varphi=\pm 1$$



$$x^2+y^2+z^2=4 \Rightarrow \rho^2=4 \Rightarrow \rho=\pm 2 \text{ tj. } \underset{\text{udaljenost tačke}}{\rho=2}$$

$$\Omega:\begin{cases} z^2=x^2+y^2 \\ x^2+y^2+z^2=4 \end{cases} \xrightarrow{\text{transformacije}} \Omega':\begin{cases} tg\varphi=\pm 1 \\ \rho=2 \end{cases}$$

Odredimo granice za drugo tijelo $\Omega'_{V_2}: \begin{cases} 0 \le \rho \le 2 \\ 0 \le \theta \le 2\pi \\ \frac{\pi}{4} \le \varphi \le \frac{3\pi}{4} \end{cases}$

$$V_2 = \iiint_{\Omega'_{V_2}} \rho^2 \sin\varphi \, d\theta \, d\varphi \, d\rho = \int_0^{2\pi} d\theta \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin\varphi \, d\varphi \int_0^2 \rho^2 \, d\rho =$$

$$= 2\pi \cdot (-\cos\varphi)\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cdot \frac{1}{3}\rho^3\Big|_0^2 = 2\pi\left(-\cos\frac{3\pi}{4} + \cos\frac{\pi}{4}\right) \cdot \frac{8}{3} =$$

$$= 2\pi\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) \cdot \frac{8}{3} = 2\pi\sqrt{2} \cdot \frac{8}{3} = \frac{16\pi\sqrt{2}}{3}$$ traženo rješenje

Zapreminu V_1 sad možemo odrediti na dva načina

I način:

$$V = V_1 + V_2 = \frac{32\pi}{3} \Rightarrow V_1 = \frac{32\pi}{3} - V_2 = \frac{32\pi}{3} - \frac{16\pi\sqrt{2}}{3}$$

$$V_1 = \frac{16\pi}{3}(2 - \sqrt{2})$$ traženo rješenje

II način:

Ako uzmemo u obzir simetričnost date oblasti Ω' u odnosu na xOy-ravan, možemo računati polovinu zapremine V_1 za $z \ge 0$ i tada bi trebalo odabrati sljedeće granice $\Omega'_{V_1}: \begin{cases} 0 \le \rho \le 2 \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \frac{\pi}{4} \end{cases}$

$$V_1 = \iiint_{\Omega'_{V_1}} \rho^2 \sin\varphi \, d\theta \, d\varphi \, d\rho$$

$$\frac{1}{2}V_1 = \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\varphi \, d\varphi \int_0^2 \rho^2 \, d\rho = 2\pi(-\cos\varphi)\Big|_0^{\pi/4} \cdot \frac{\rho^3}{3}\Big|_0^2 =$$

$$= 2\pi\left(1 - \cos\frac{\pi}{4}\right) \cdot \frac{8}{3} = 2\pi\left(1 - \frac{\sqrt{2}}{2}\right) \cdot \frac{8}{3}.$$

$$\Rightarrow V_1 = 4\pi\left(1 - \frac{\sqrt{2}}{2}\right) \cdot \frac{8}{3} = 4\pi \cdot \frac{2 - \sqrt{2}}{2} \cdot \frac{8}{3} = \frac{16\pi}{3}(2 - \sqrt{2}).$$